Notes#12 Volumes of Revolution

Volume of revolution is the last topic involving integration we will look at.

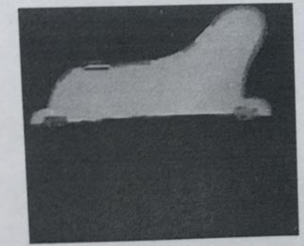
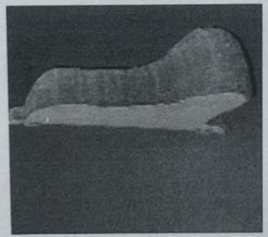
A solid of revolution is a three-dimensional object obtained by rotating a function *f(x)* in the plane about a line in the plane. The volume of this solid may be calculated by means of integration.

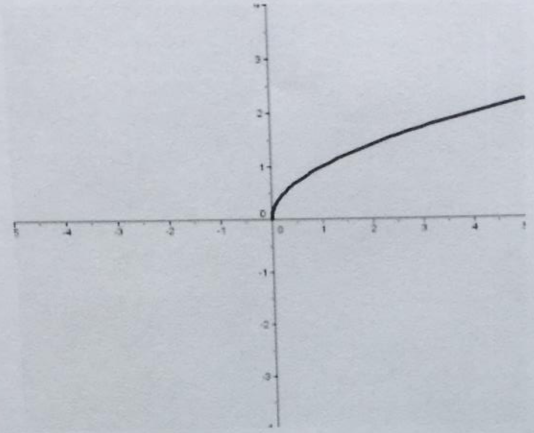
Great explanation, checkout https://www.youtube.com/watch?v=btGaOTXxXs8

Volume of Revolution

We must be able to visualize these solids of revolution...

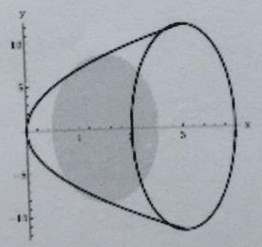
This simple example may help:

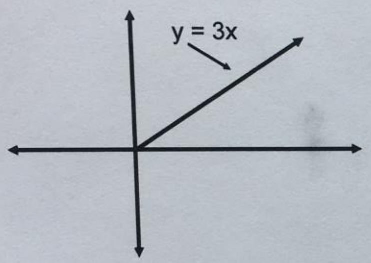



$y = \sqrt{x}$

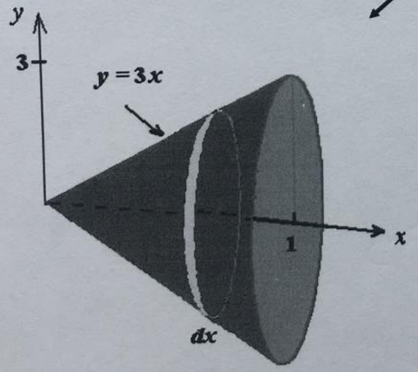
rotate about the x-axis

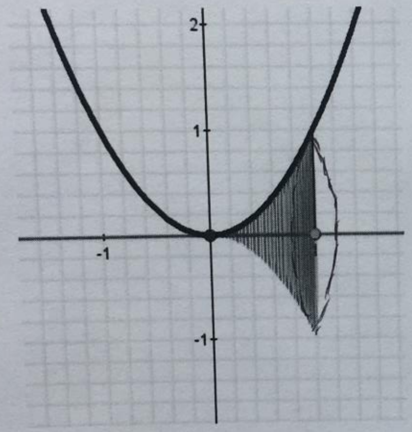


The function y = 3x rotated about the the x-axis.



the function y = 3x rotated about the x-axis

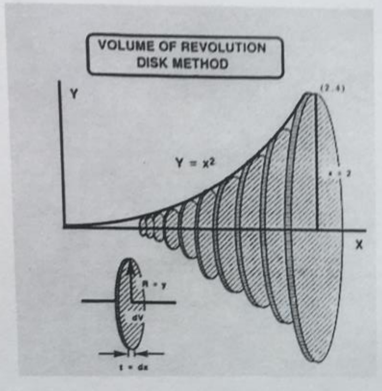


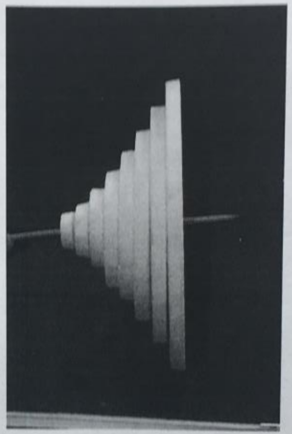


The area under the curve of $y = x^2$ from 0 to 1 rotated about the x-axis.

How do we find the volume?

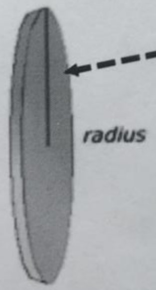
Instead of finding area with an infinite # of rectangles, we find the volume with an infinite # of disks





Cross-sections are cylindrical disks

- How do we find the volume of each of these disks?



This is a right cylinder

volume of a cylinder is $\pi r^2 h$

$$V = \pi\int_a^b [f(x)]^2 dx$$

V = volume of the solid
a is smallest value of x of f(x)
b is the largest value of x of f(x)
f(x) is the value of radius of the disc
dx is the value of the height of the disc

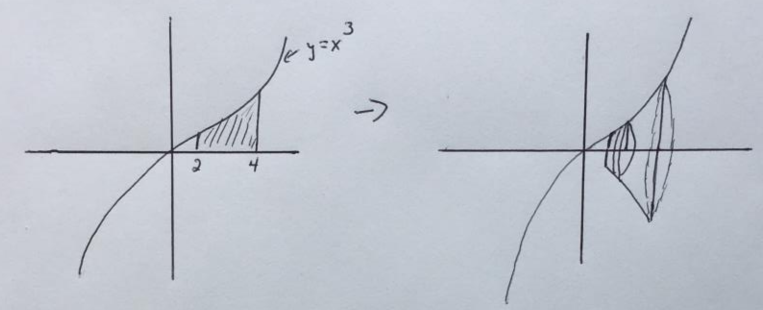
$$V = \int_a^b \pi r^2 h$$

$$V = \pi \int_a^b r^2 h$$

$$V = \pi \int_a^b (f(x))^2 dx$$

$$V = \pi \int_a^b (f(x))^2 dx$$

Determine the volume of the region obtained by rotating the area between $y = x^3$ and the x axis from x=2 to x=4 about the x axis.



$$V = \int \pi r^2 h$$

$$= \pi \int_2^4 (x^3)^2 dx$$

$$= \pi \int_2^4 x^6 dx$$

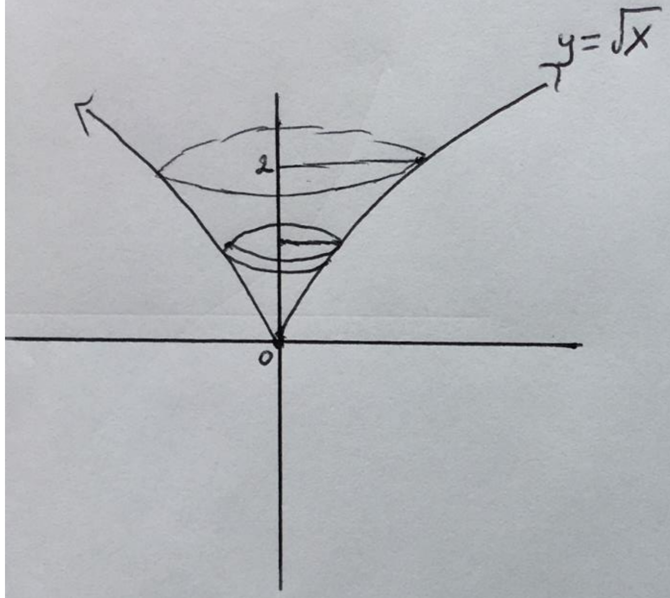
$$= \pi \left[\frac{x^7}{7} \right]_2^4$$

$$= \pi \left[\left(\frac{4^7}{7} \right) - \left(\frac{2^7}{7} \right) \right]$$

$$= \pi \left[\frac{16384}{7} - \frac{128}{7} \right]$$

$$= \frac{16256}{7} \pi u^3$$

Determine the volume from rotating theregion bound by $y = \sqrt{x}$ and the y-axis about the y-axis. (from y=0 to y=2)



$(y)^2 = (\sqrt{x})^2$

$y^2 = x$

$V = \int_0^2 \pi (y^2)^2 \, dy$

$= \pi \int_0^2 y^4 \, dy$

$= \pi \left[\frac{y^5}{5} \right]_0^2$

$= \pi \left[\left(\frac{2^5}{5} \right) - \frac{(0)^5}{5} \right]$

$= \frac{32}{5} \pi \, u^3$

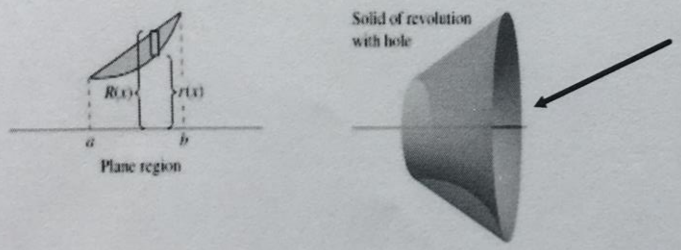
Another method used to find volume of a solid is the " **Washer Method"**

This method is required when the shape that you produce is not solid all the way through.

Like a vase, bowl, donut etc..

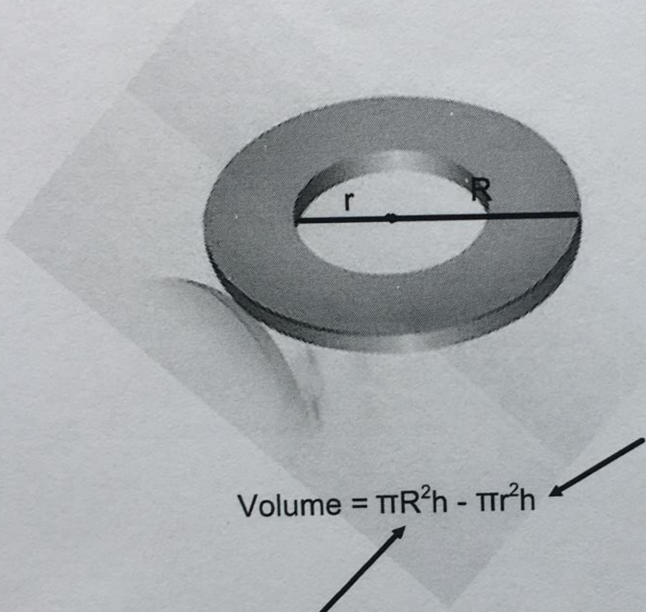
Ex.

To see how this concept can be used to find the volume of a solid of revolution, consider a region bounded by an **outer radius** $R(x)$ and an **inner radius** $r(x)$, as shown in Figure 7.19.



open space inside

Volume of a Washer



volume of empty center

Volume = $\pi R^2 h - \pi r^2 h$

volume of entire disk

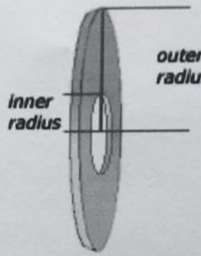
$$V = \int \pi R^2 h - \pi r^2 h$$

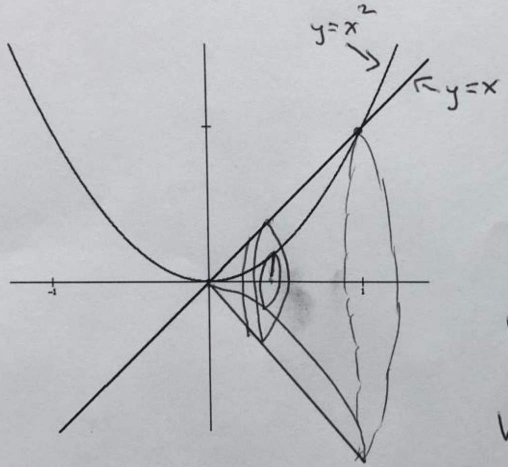
$$V = \pi \int (R^2 - r^2) h$$

Washer Method

Example:

Sketch the region bound by the curves $y = x^2$ and $y = x$, and determine the volume of the solid generated by rotating this region about...
(i) the x-axis
(ii) the y-axis





$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

$V = \pi \int_0^1 (R^2 - r^2)\, dx$

$V = \pi \int_0^1 (x)^2 - (x^2)^2\, dx$

$V = \pi \int_0^1 x^2 - x^4\, dx$

$= \pi\left[\left[\frac{(1)^3}{3} - \frac{(1)^5}{5}\right] - \left[\frac{0}{3} - \frac{0}{5}\right]\right] = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$

$= \pi\left[\frac{1}{3} - \frac{1}{5}\right]$

$= \pi\left[\frac{5}{15} - \frac{3}{15}\right]$

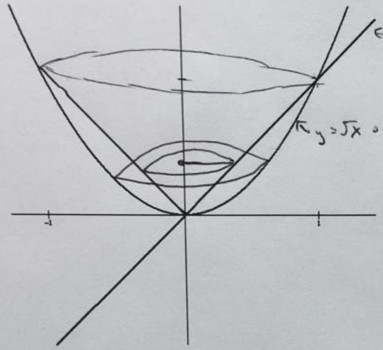
$= \frac{2}{15}\pi\ u^3$

Rotate about the y-axis

$y = x^2$ $\qquad$ $x = y$

$\sqrt{y} = x$

$x = y^{\frac{1}{2}}$



$$V = \pi \int (R^2 - r^2)\, h$$

$$V = \pi \int_0^1 (y^{\frac{1}{2}})^2 - (y)^2 \, dy$$

$$V = \pi \int_0^1 y - y^2 \, dy$$

$$= \pi \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1$$

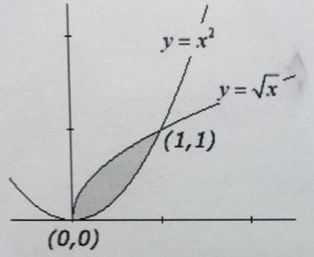
$$= \pi \left[\frac{1}{2} - \frac{1}{3} \right] - 0$$

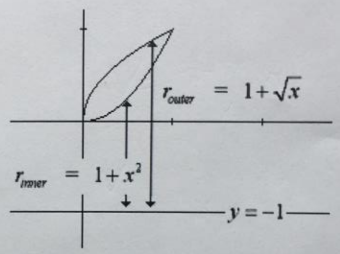
$$= \pi \left[\frac{3}{6} - \frac{2}{6} \right]$$

$$= \frac{\pi}{6} u^3$$

4) *Find the volume of the solid obtained by rotating the region bounded by the curve $y = x^2$ and $y = \sqrt{x}$ about the line $y = -1$.*

Solution The volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = x^2$ about the line $y = -1$ is equal to $\frac{29\pi}{30}$ units cubed.





$V = \pi \int (R^2 - r^2) h$

$= \pi \int_0^1 (1+\sqrt{x})^2 - (1+x^2)^2 \, dx$

$= \pi \int_0^1 (1+2\sqrt{x}+x) - (1+2x^2+x^4) \, dx$

$= \pi \int_0^1 1 + 2\sqrt{x} + x - 1 - 2x^2 - x^4 \, dx$

$= \pi \int_0^1 2\sqrt{x} + x - 2x^2 - x^4 \, dx$

$= \pi \left[\frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2} - \frac{2}{3}x^3 - \frac{x^5}{5} \right]_0^1$

$= \pi \left[\frac{4}{3} + \frac{1}{2} - \frac{2}{3} - \frac{1}{5} \right] - 0$

$= \pi \left[\frac{40}{30} + \frac{15}{30} - \frac{20}{30} - \frac{6}{30} \right]$

$\frac{29}{30}\pi \, u^3$