Notes#9 Integration By Parts

Integration By Parts

The integration by parts formula is a product rule for integration.

Product rule

$$d(uv) = u \cdot dv + v \cdot du$$

integrate

$$\int d(uv) = \int udv + \int vdu$$

$$uv = \int udv + \int vdu$$

$$uv - \int vdu = \int udv$$

**** Formula

$$\int udv \overset{**}{=} \boxed{uv - \int vdu}$$

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

use when 2 parts are <u>totally unrelated</u>!

What this means is a hint to use the last method u- substitution you look for a function and part of its derivative in the same question. With integration by parts you break the question into two parts because there is no relationship between them.One piece is not the derivative of the other.

One part will be called u , the other part will be dv.

** You will take the <u>derivative</u> of the u equation and the <u>integral</u> of the dv equation

<u>How you pick what u will be</u>

Choose u in this order : LIPET

Logs ex. $\ln x$

Inverse Trig. ex. $\tan^{-1} x$

Polynomial ex. x, x^2, x^3

Exponential ex. e^x

Trig ex. $\sin\theta$

Ex. $\int xe^x \, dx$

2 parts to look at, x and e^x. The derivative of x is 1 and the derivative of e^x is e^x so one part is not the derivative of the other (no u-substitution). No relation between the 2 parts.

Picking which part is $\underline{u}$ → Using LIPET, polynomial (x) comes before exponential (e^x). So x will be u and dv will be e^x.

Take derivative of this equation → $u = x$

$du = dx$

$dv = e^x \, dx$ ← take integral of this equation

$v = e^x$

Fill into formula

$$= uv - \int v \, du$$

$$= xe^x - \int e^x \, dx$$

$$\boxed{= xe^x - e^x + C}$$

← Answer check: If you took the derivative you get the original (xe^x)

Ex. $\int x \cos 3x \, dx$

$u = x \qquad dv = \cos 3x \, dx$

$du = dx \qquad v = \frac{\sin 3x}{3}$

$= uv - \int v \, du$

$= x \frac{\sin 3x}{3} - \frac{1}{3} \int \sin 3x \, dx$

$= \frac{1}{3} x \sin 3x - \frac{1}{3} \cdot \frac{-\cos 3x}{3} + C$

$= \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + C$

$\boxed{= \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + C}$

Ex. $\int \ln x \, dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$= uv - \int v \, du$

$= \ln x (x) - \int x \cdot \frac{1}{x} dx$

$= x \ln x - \int \frac{x}{x} dx$

$= x \ln x - \int 1 \, dx$

$\boxed{= x \ln x - x + C}$

Ex. $\int x^7 \ln 5x \, dx$

$u = \ln 5x \qquad dv = x^7 dx$

$du = \frac{1}{5x}(5)\,dx \qquad v = \frac{x^8}{8}$

$du = \frac{1}{x}\,dx$

$= uv - \int v\,du$

$= \ln 5x\left(\frac{x^8}{8}\right) - \frac{1}{8}\int x^8 \cdot \frac{1}{x}\,dx$

$= \frac{1}{8}x^8 \ln 5x - \frac{1}{8}\int x^7\,dx$

$= \frac{1}{8}x^8 \ln 5x - \frac{1}{8} \cdot \frac{x^8}{8} + C$

$$\boxed{= \frac{1}{8}x^8 \ln 5x - \frac{1}{64}x^8 + C}$$

* Integration by Parts Twice

Ex. $\int x^2 e^x \, dx$

$u = x^2 \qquad dv = e^x \, dx$

$du = 2x \, dx \qquad v = e^x$

$= uv - \int v \, du$

$= x^2 e^x - 2\int e^x x \, dx$

Integration by parts again

$u = x \qquad dv = e^x \, dx$

$du = dx \qquad v = e^x$

$= x^2 e^x - 2[uv - \int v \, du]$

$= x^2 e^x - 2[xe^x - \int e^x dx]$

$= x^2 e^x - 2[xe^x - e^x]$

$\boxed{= x^2 e^x - 2xe^x + 2e^x + C}$

* TWICE

Ex. $\int x^2 \sin 3x \, dx$

$u = x^2 \qquad dv = \sin 3x \, dx$

$du = 2x\,dx \qquad v = -\dfrac{\cos 3x}{3}$

$= uv - \int v\,du$

$= x^2\left(-\dfrac{\cos 3x}{3}\right) + \dfrac{2}{3}\int \cos 3x \cdot x\,dx$

$u = x \qquad dv = \cos 3x\,dx$

$du = dx \qquad v = \dfrac{\sin 3x}{3}$

$= -\dfrac{1}{3}x^2\cos 3x + \dfrac{2}{3}\int x\cos 3x\,dx$

$= -\dfrac{1}{3}x^2\cos 3x + \dfrac{2}{3}\left[uv - \int v\,du\right]$

$= -\dfrac{1}{3}x^2\cos 3x + \dfrac{2}{3}\left[x\left(\dfrac{\sin 3x}{3}\right) - \dfrac{1}{3}\int \sin 3x\,dx\right]$

$= -\dfrac{1}{3}x^2\cos 3x + \dfrac{2}{3}\left[\dfrac{1}{3}x\sin 3x - \dfrac{1}{3}\cdot\left(-\dfrac{\cos 3x}{3}\right) + C\right]$

$= -\dfrac{1}{3}x^2\cos 3x + \dfrac{2}{3}\left[\dfrac{1}{3}x\sin 3x + \dfrac{1}{9}\cos 3x + C\right]$

$$\boxed{= -\dfrac{1}{3}x^2\cos 3x + \dfrac{2}{9}x\sin 3x + \dfrac{2}{27}\cos 3x + C}$$

Ex. $\int x^2 \ln x \, dx$

$u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^3}{3}$

$= uv - \int v \, du$

$= \ln x \left(\frac{x^3}{3}\right) - \frac{1}{3}\int x^3 \cdot \frac{1}{x} dx$

$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 dx$

$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + C$

$\boxed{= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C}$